

COMPOSITION OPERATORS BETWEEN ANALYTIC CAMPANATO SPACES

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ABSTRACT. This note characterizes both boundedness and compactness of a composition operator between any two analytic Campanato spaces on the unit complex disk.

1. INTRODUCTION

On the basis of the works: [15], [5], [6, 7], [8], [18], [19], [17], [1], [9, 10], [12], [3], [20, 21, 22], [13, 14] and [2], we consider an unsolved fundamental problem in the function-theoretic operator theory, i.e., the so-called composition operator question for the analytic Campanato spaces:

Question 1. *Let ϕ be an analytic self-map of \mathbb{D} and $-\infty < p, q < \infty$. What finite (resp. vanishing) property must ϕ have in order that C_ϕ is bounded (resp. compact) between $C\mathcal{A}_p$ and $C\mathcal{A}_q$?*

In the above and below, \mathbb{D} and \mathbb{T} respectively represent the unit disk and the unit circle in the finite complex plane \mathbb{C} , $C_\phi f = f \circ \phi$ is the composition of an analytic function f on \mathbb{D} with ϕ , and for $p \in (-\infty, \infty)$, and $C\mathcal{A}_p$ denotes the so-called Campanato space of all analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ with radial boundary values f on \mathbb{T} satisfying

$$\|f\|_{C\mathcal{A}_p} = \sup_{I \subseteq \mathbb{T}} \sqrt{|I|^{-p} \int_I |f(\xi) - f_I|^2 |d\xi|} < \infty$$

where the supremum is taken over all sub-arcs $I \subseteq \mathbb{T}$ with $|I|$ being their arc-lengths, and

$$|d\xi| = |de^{i\theta}| = d\theta; \quad f_I = |I|^{-1} \int_I f(\xi) |d\xi|.$$

Needless to say, $\|\cdot\|_{C\mathcal{A}_p}$ cannot distinguish between any two $C\mathcal{A}_p$ functions differing by a constant, but $|f(0)| + \|\cdot\|_{C\mathcal{A}_p}$ defines a norm so that $C\mathcal{A}_p$

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is a Banach space. Here, it is perhaps appropriate to mention the following table which helps us get a better understanding of the structure of $C\mathcal{A}_p$ (see, e.g. [4, pp. 67-75] and [22, p. 52]):

Index p	Analytic Campanato Space $C\mathcal{A}_p$
$p \in (-\infty, 0]$	Analytic Hardy space \mathcal{H}^2
$p \in (0, 1)$	Holomorphic Morrey space $\mathcal{H}^{2,p}$
$p = 1$	Analytic John-Nirenberg space $\mathcal{BMO}\mathcal{A}$
$p \in (1, 3]$	Analytic Lipschitz space $\mathcal{A}_{\frac{p-1}{2}}$
$p \in (3, \infty)$	Complex constant space \mathbb{C}

An answer to the boundedness part of Question 1 is the following result.

Theorem 1. *Let ϕ be an analytic self-map of \mathbb{D} and $(p, q) \in [0, 2) \times [0, 2)$. Then $C_\phi : C\mathcal{A}_p \mapsto C\mathcal{A}_q$ is bounded if and only if*

$$(1) \quad \sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|_2^2 < \infty,$$

where

$$\sigma_b(z) = \frac{b - z}{1 - \bar{b}z} \quad \& \quad \|f\|_2 = \sqrt{\int_{\mathbb{T}} |f(\xi)|^2 |d\xi|}.$$

It should be pointed out that (1) is not always true - in fact, we have the following consequence whose (i) with $p = q \in \{0, 1\}$ and (ii) are well-known; see e.g. [9, 12, 13, 14, 15, 20].

Corollary 1. *Let ϕ be an analytic self-map of \mathbb{D} . For $f \in \mathcal{H}^2$ and $p \in [0, 2)$ set*

$$\|f\|_{C\mathcal{A}_p,*} = \sup_{a \in \mathbb{D}} (1 - |a|^2)^{\frac{1-p}{2}} \|f \circ \sigma_a - f(a)\|_2.$$

(i) *If $p \in [0, 1]$ then $C_\phi : C\mathcal{A}_p \mapsto C\mathcal{A}_p$ is always bounded with*

$$(2) \quad \|C_\phi f\|_{C\mathcal{A}_p,*} \leq \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|} \right)^{\frac{1-p}{2}} \|f\|_{C\mathcal{A}_p,*}.$$

(ii) *If $p \in (1, 2)$ then $C_\phi : C\mathcal{A}_p \mapsto C\mathcal{A}_p$ is bounded when and only when*

$$(3) \quad \sup_{a \in \mathbb{D}} \left(\frac{1 - |a|^2}{1 - |\phi(a)|^2} \right)^{\frac{3-p}{2}} |\phi'(a)| < \infty.$$

Below is a partial answer to the compactness part of Question 1.

Theorem 2. *Let ϕ be an analytic self-map of \mathbb{D} and $(p, q) \in [0, 2) \times [0, 2)$. If $C_\phi : C\mathcal{A}_p \mapsto C\mathcal{A}_q$ is compact then (1) holds and*

$$(4) \quad \lim_{|\phi(a)| \rightarrow 1} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|_2^2 = 0.$$

Conversely, if (1) holds and (4) is valid for $(p, q) \in [0, 2) \times [1, 1] \cup (1, 2) \times [0, 2)$ then $C_\phi : C\mathcal{A}_p \mapsto C\mathcal{A}_q$ is compact.

Theorem 2 covers the corresponding $BMO\mathcal{A}$ -results in [15, 19, 8], but also it derives the following assertion extending the known one in [12, 10, 20].

Corollary 2. *Let ϕ be an analytic self-map of \mathbb{D} and $p \in [0, 2)$. If $C_\phi : C\mathcal{A}_p \mapsto C\mathcal{A}_p$ is compact then (3) holds and*

$$(5) \quad \lim_{|\phi(a)| \rightarrow 1} \left(\frac{1 - |a|^2}{1 - |\phi(a)|^2} \right)^{\frac{3-p}{2}} |\phi'(a)| = 0.$$

Conversely, if (3) holds and (5) is valid for $p \in (1, 2)$ then $C_\phi : C\mathcal{A}_p \mapsto C\mathcal{A}_p$ is compact.

Conjecture 1. *The converse part of Theorem 2 still holds for $(p, q) \in [0, 2) \times [0, 2) \setminus ([0, 2) \times [1, 1] \cup (1, 2) \times [0, 2))$.*

Notation: From now on, $X \lesssim Y$, $X \gtrsim Y$, and $X \approx Y$ represent that there exists a constant $\kappa > 0$ such that $X \leq \kappa Y$, $X \geq \kappa Y$, and $\kappa^{-1}Y \leq X \leq \kappa Y$, respectively. In addition, dm stands for two dimensional Lebesgue measure.

2. BOUNDEDNESS

In order to prove Theorem 1 and Corollary 1, we need two lemmas.

Lemma 1. *Let $p \in [0, 2)$ and $f \in \mathcal{H}^2$. Then $f \in C\mathcal{A}_p$ if and only if $\|f\|_{C\mathcal{A}_p, *} < \infty$.*

Proof. *Case 1:* $p = 0$. This is trivial.

Case 2: $p \in (0, 1]$. This situation can be verified by [22, Theorem 3.2.1] and the well-known Hardy-Littlewood identity for $f \in \mathcal{H}^2$:

$$(6) \quad \pi^{-1} \int_{\mathbb{D}} |f'(z)|^2 (-\ln |z|^2) dm(z) = (2\pi)^{-1} \int_{\mathbb{T}} |f(\xi) - f(0)|^2 |d\xi|.$$

Case 3: $p \in (1, 2)$. Let $g = f \circ \sigma_a - f(a)$. Then

$$(7) \quad (1 - |a|^2)|f'(a)| = |g'(0)| \leq (2\pi)^{-1/2} \|g\|_2 = (2\pi)^{-1/2} \|f \circ \sigma_a - f(a)\|_2.$$

If $f \in C\mathcal{A}_p$ then an application of (7) yields

$$(8) \quad \sup_{a \in \mathbb{D}} (1 - |a|^2)^{\frac{3-p}{2}} |f'(a)| < \infty$$

and consequently, $f \in \mathcal{A}_{\frac{p-1}{2}}$, as desired. Conversely, if $f \in \mathcal{A}_{\frac{p-1}{2}}$ then

$$A = \sup_{\xi_1 \neq \xi_2 \text{ in } \mathbb{D} \cup \mathbb{T}} \frac{|f(\xi_1) - f(\xi_2)|}{|\xi_1 - \xi_2|^{\frac{p-1}{2}}} < \infty.$$

This, along with $p \in (1, 2)$ and [23, p. 63, Ex. 8], gives

$$\begin{aligned}
& \|f\|_{C\mathcal{A}_{p,*}}^2 \\
&= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-p} \int_{\mathbb{T}} |f \circ \sigma_a(\xi) - f(a)|^2 |d\xi| \\
&\lesssim A^2 \sup_{a \in \mathbb{D}} \int_{\mathbb{T}} \left(\frac{1 - |a|^2}{|\sigma_a(\xi) - a|} \right)^{1-p} |d\xi| \\
&\approx A^2 \sup_{a \in \mathbb{D}} \int_{\mathbb{T}} \frac{(1 - |a|^2)^{2-p}}{|1 - \bar{a}\eta|^{3-p}} |d\eta| \\
&\lesssim A^2.
\end{aligned}$$

□

Lemma 2. For $p \in [0, 2)$ let $f_b(z) = (1 - |b|^2)^{\frac{1+p}{2}} / (1 - \bar{b}z)$. Then f_b is uniformly bounded in $C\mathcal{A}_p$, i.e., $\sup_{b \in \mathbb{D}} \|f_b\|_{C\mathcal{A}_{p,*}} < \infty$.

Proof. Using [11, Lemma 2.5], we get the following estimate:

$$\begin{aligned}
& \mathbf{B} \\
&= \int_{\mathbb{D}} |f'_b(z)|^2 (1 - |\sigma_a(z)|^2) dm(z) \\
&= (|b|(1 - |b|^2)^{\frac{1+p}{2}})^2 (1 - |a|^2) \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - \bar{a}z|^2 |1 - \bar{b}z|^4} dm(z) \\
&\lesssim \frac{(|b|(1 - |b|^2)^{\frac{1+p}{2}})^2 (1 - |a|^2)}{(1 - |b|^2) |1 - \bar{a}b|^2}.
\end{aligned}$$

Choosing $a = \sigma_b(c)$, we utilize $1 - |z| \lesssim -\ln |z|$ to obtain that if $p \in [0, 2)$ then

$$\begin{aligned}
& \|f_b\|_{C\mathcal{A}_{p,*}}^2 \\
&\lesssim \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-p} \mathbf{B} \\
&\lesssim \sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{2-p} (1 - |b|^2)^p}{|1 - \bar{a}b|^2} \\
&= \sup_{c \in \mathbb{D}} \left(\frac{1 - |c|^2}{|1 - \bar{b}c|} \right)^{2-p} |1 - \bar{b}c|^p \\
&\leq 2^2,
\end{aligned}$$

as desired. □

Proof of Theorem 1. Using [6, Proposition 2.3], we have that if $g(0) = 0 = \psi(0)$, $g \in \mathcal{H}^2$, and ψ is an analytic self-map of \mathbb{D} , then

$$(9) \quad \|g \circ \psi\|_2 \lesssim \|g\|_2 \|\psi\|_2.$$

Setting

$$g_a = f \circ \sigma_{\phi(a)} - f \circ \phi(a) \quad \& \quad \psi_a = \sigma_{\phi(a)} \circ \phi \circ \sigma_a,$$

we get

$$g_a \circ \psi_a = f \circ \phi \circ \sigma_a - f \circ \phi(a).$$

As a consequence of Lemma 1 and (9), we find that if (1) is valid then

$$\begin{aligned} & \|C_\phi f\|_{\mathcal{CA}_q,*}^2 \\ &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-q} \|f \circ \phi \circ \sigma_a - f \circ \phi(a)\|_2^2 \\ &\lesssim \sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} ((1 - |\phi(a)|^2)^{1-p} \|g_a\|_2^2) \|\psi_a\|_2^2 \\ &\lesssim \|f\|_{\mathcal{CA}_p,*}^2 \sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\psi_a\|_2^2, \end{aligned}$$

and consequently, C_ϕ exists as a bounded operator from \mathcal{CA}_p into \mathcal{CA}_q .

For the “only-if” part, recall the so-called Nevanlinna counting function of ϕ :

$$N(\phi, w) = \sum_{z: \phi(z)=w} \ln |z|^{-1} \quad \forall \quad w \in \mathbb{D} \setminus \{\phi(0)\}$$

and the associated change of variable formula:

$$(10) \quad \int_{\mathbb{D}} |(C_\phi f)'(z)|^2 \ln |z|^{-1} dm(z) = \int_{\mathbb{D}} |f'(w)|^2 N(\phi, w) dm(w) \quad \forall f \in \mathcal{H}^2.$$

A combination of (10) and (6) gives that if $b = \phi(a)$ then

$$(11) \quad \|\sigma_b \circ \phi \circ \sigma_a\|_2^2 = 4 \int_{\mathbb{D}} N(\sigma_b \circ \phi \circ \sigma_a, z) dm(z).$$

Now, if $C_\phi : \mathcal{CA}_p \mapsto \mathcal{CA}_q$ is bounded, then the test function f_b in Lemma 2 is used to imply

$$\mathbf{C} = \sup_{a,b \in \mathbb{D}} (1 - |a|^2)^{1-q} \int_{\mathbb{D}} |(f_b \circ \phi \circ \sigma_a)'(z)|^2 \ln |z|^{-1} dm(z) < \infty.$$

and consequently,

C

$$\begin{aligned}
&\gtrsim \sup_{a,b \in \mathbb{D}} (1 - |a|^2)^{1-q} |b|^2 (1 - |b|^2)^{p-1} \int_{\mathbb{D}} |\sigma'_b(z)|^2 N(\phi \circ \sigma_a, z) dm(z) \\
&\gtrsim \sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q} |\phi(a)|^2}{(1 - |\phi(a)|^2)^{1-p}} \int_{\mathbb{D}} N(\sigma_{\phi(a)} \circ \phi \circ \sigma_a, z) dm(z) \\
&\gtrsim \sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q} |\phi(a)|^2}{(1 - |\phi(a)|^2)^{1-p}} \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|_2^2 \\
&\gtrsim s^2 \sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|_2^2
\end{aligned}$$

as $|\phi(a)| > s \in (0, 1)$. Note also that the identity map $f(z) = z$ is an element of $C\mathcal{A}_p$. Thus, boundedness of $C_\phi : C\mathcal{A}_p \mapsto C\mathcal{A}_q$ ensures $\|\phi\|_{C\mathcal{A}_q,*} < \infty$, and consequently, if $|\phi(a)| \leq s < 1$ then

$$\begin{aligned}
&\sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|_2^2 \\
&\lesssim (1 + (1 - s)^{p-1}) \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-q} \int_{\mathbb{T}} \left| \frac{\phi(a) - \phi \circ \sigma_a(\xi)}{1 - \overline{\phi(a)} \phi \circ \sigma_a(\xi)} \right|^2 |d\xi| \\
&\lesssim \left(\frac{1 + (1 - s)^{p-1}}{(1 - s)^2} \right) \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-q} \|\phi(a) - \phi \circ \sigma_a\|_2^2 \\
&\approx \left(\frac{1 + (1 - s)^{p-1}}{(1 - s)^2} \right) \|\phi\|_{C\mathcal{A}_q,*}^2.
\end{aligned}$$

The above estimates imply (1). □

Proof of Corollary 1. (i) Under $p \in [0, 1]$, we use the Schwarz lemma for $\sigma_{\phi(0)} \circ \phi$ to deduce that (1) holds for $p = q \in [0, 1]$, and so that C_ϕ is bounded on $C\mathcal{A}_p$ due to Theorem 1. To reach (2), let us begin with the case $\phi(0) = 0$. According to the setting in the argument for Theorem 1, the well-known Littlewood subordination principle and Schwarz's lemma for ϕ , we have

$$\begin{aligned}
&\|C_\phi f\|_{C\mathcal{A}_p,*}^2 \\
&= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-p} \|g_a \circ \psi_a\|_2^2 \\
&\leq \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-p} \|g_a\|_2^2 \\
&\leq \|f\|_{C\mathcal{A}_p,*}^2 \sup_{a \in \mathbb{D}} \left(\frac{1 - |a|^2}{1 - |\phi(a)|^2} \right)^{1-p} \\
&\leq \|f\|_{C\mathcal{A}_p,*}^2.
\end{aligned}$$

Next, for the general case let

$$\begin{cases} \psi = \sigma_{\phi(0)} \circ \phi; \\ \lambda = \frac{ab-1}{1-ab}; \\ b = \phi(0); \\ c = \sigma_a(b). \end{cases}$$

Then $\psi(0) = 0$ and thus

$$\begin{aligned} & \|C_{\sigma_b} f\|_{C\mathcal{A}_{p,*}}^2 \\ &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-p} \|f(\lambda\sigma_c) - f(\lambda c)\|_2^2 \\ &\leq \|f\|_{C\mathcal{A}_{p,*}}^2 \sup_{a \in \mathbb{D}} \left(\frac{1 - |a|^2}{1 - |c|^2} \right)^{1-p} \\ &\leq \|f\|_{C\mathcal{A}_{p,*}}^2 \left(\frac{1 + |b|}{1 - |b|} \right)^{1-p}. \end{aligned}$$

Using the previous estimates, we get

$$\|C_\phi f\|_{C\mathcal{A}_{p,*}}^2 = \|f \circ \sigma_b \circ \psi\|_{C\mathcal{A}_{p,*}}^2 \leq \|f \circ \sigma_b\|_{C\mathcal{A}_{p,*}}^2 \leq \|f\|_{C\mathcal{A}_{p,*}}^2 \left(\frac{1 + |b|}{1 - |b|} \right)^{1-p},$$

whence reaching (2).

(ii) Suppose $p \in (1, 2)$. Then (7) yields

$$\sup_{a \in \mathbb{D}} \left(\frac{1 - |a|^2}{1 - |\phi(a)|^2} \right)^{3-p} |\phi'(a)|^2 \leq \sup_{a \in \mathbb{D}} \left(\frac{1 - |a|^2}{1 - |\phi(a)|^2} \right)^{1-p} \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|_2^2,$$

so, if C_ϕ is bounded on $C\mathcal{A}_p$ then (1) holds with $p = q$ due to Theorem 1, and hence (3) holds. Conversely, if (3) is true, then C_ϕ is bounded on $\mathcal{A}_{\frac{p-1}{2}}$ (cf. [9, Theorem A]) and hence bounded on $C\mathcal{A}_p$. \square

3. COMPACTNESS

The arguments for Theorem 2 and Corollary 2 depend on the two basic facts below.

Lemma 3. *Let $p \in (0, 2)$ and $f \in C\mathcal{A}_p$ with $f(0) = 0$. Then*

$$\int_{\mathbb{T}} |f(\xi)|^4 |d\xi| \lesssim \begin{cases} \|f\|_{C\mathcal{A}_{p,*}}^2 \|f\|_2^2 & \text{for } p \in [1, 2); \\ \|f\|_{C\mathcal{A}_{p,*}}^2 \int_0^\infty t H_\infty^p(\{\xi \in \mathbb{T} : |f(\xi)| > t\}) dt & \text{for } p \in (0, 1], \end{cases}$$

where $H_\infty^p(E) = \inf_{E \subseteq \cup_j I_j} \sum_j |I_j|^p$ is p -dimensional Hausdorff capacity of $E \subseteq \mathbb{T}$ - the infimum is taken over all arc coverings $\cup_j I_j \supseteq E$.

Proof. Let $d\mu = |f'(z)|^2(1 - |z|^2)dm(z)$. From $f \in C\mathcal{A}_p$ it follows that μ is a p -Carleson measure on \mathbb{D} - in other words -

$$\|\mu\|_{CM_p} = \sup_{I \subseteq \mathbb{T}} |I|^{-p} \mu(S(I)) \lesssim \|f\|_{C\mathcal{A}_p, *}^2,$$

where $S(I) = \{z = re^{i\theta} \in \mathbb{D} : 1 - |I|/(2\pi) \leq r < 1 \text{ \& } |\theta - \theta_I| \leq |I|/2\}$ is the Carleson box based on the arc $I \subseteq \mathbb{T}$ taking θ_I as its center. In fact, if $a = (1 - |I|/(2\pi))e^{i(\theta_I + |I|/4)}$ then a simple computation, along with (6) and $-\ln|z| \approx 1 - |z|^2$ as $|z| \geq 2^{-1}$ as well as Lemma 1, gives

$$\begin{aligned} & |I|^{-p} \mu(S(I)) \\ & \lesssim (1 - |a|^2)^{1-p} \int_{S(I)} |f'(z)|^2 (1 - |\sigma_a(z)|^2) dm(z) \\ & \lesssim (1 - |a|^2)^{1-p} \|f \circ \sigma_a - f(a)\|_2^2 \\ & \lesssim \|f\|_{C\mathcal{A}_p, *}^2. \end{aligned}$$

In particular, when $p \in (1, 2)$, μ is also 1-Carleson measure with $\|\mu\|_{CM_1} \lesssim \|\mu\|_{CM_p}$. According to [22, p. 79, Theorem 4.1.4], we have

$$\int_{\mathbb{D}} |f|^2 d\mu \lesssim \|\mu\|_{CM_p} \begin{cases} \|f\|_2^2 & \text{for } p \in [1, 2); \\ \int_0^\infty t H_\infty^p(\{\xi \in \mathbb{T} : |f(\xi)| > t\}) dt & \text{for } p \in (0, 1]. \end{cases}$$

This last estimate, along with the following Hardy-Stein identity based estimate (cf. [22, p. 36])

$$\begin{aligned} & \int_{\mathbb{T}} |f(\xi)|^4 |d\xi| \\ & \approx \int_{\mathbb{D}} |f(z)|^2 |f'(z)|^2 (\ln|z|^{-1}) dm(z) \\ & \lesssim \int_{\mathbb{D}} |f(z)|^2 |f'(z)|^2 (1 - |z|^2) dm(z) \\ & \approx \int_{\mathbb{D}} |f|^2 d\mu, \end{aligned}$$

implies the desired estimate. \square

Lemma 4. *Let $(p, q) \in [0, 2) \times [1, 1]$. If an analytic self-map ϕ of \mathbb{D} satisfies (4), then one has*

$$\lim_{t \rightarrow 1} \sup_{|\phi(a)| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} |\{\xi \in \mathbb{T} : |\sigma_{\phi(a)} \circ \phi \circ \sigma_a(\xi)| > t\}| = 0 \quad \forall s \in (0, 1).$$

Proof. Note that

$$|\phi \circ \sigma_a| \rightarrow 1 \iff |\sigma_{\phi(a)} \circ \phi \circ \sigma_a| \rightarrow 1 \quad \text{under } |\phi(a)| \leq s.$$

So, it suffices to show that (4) implies

$$(12) \quad \lim_{t \rightarrow 1} \sup_{|\phi(a)| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} |\{\xi \in \mathbb{T} : |\phi \circ \sigma_a(\xi)| > t\}| = 0 \quad \forall s \in (0, 1).$$

Following [8], for $re^{i\theta} \in \mathbb{D}$ let

$$J(re^{i\theta}) = \{e^{it} : |t - \theta| \leq \pi(1 - r)\}.$$

Clearly, $J(re^{i\theta})$ is the sub-arc of \mathbb{T} centered at $e^{i\theta}$. Importantly, [8, Lemma 3] tells us that for any measurable set $E \subseteq \mathbb{T}$ with 1-dimensional Lebesgue measure $|E| > 0$ there exists a measurable set $F \subseteq E$ such that $|F| > 0$ and

$$(13) \quad \frac{|J(r\xi) \cap E|}{|J(r\xi)|} \geq (2^4\pi)^{-1}|E| \quad \forall r \in [0, 1) \text{ \& } \xi \in F.$$

Suppose now (4) is valid but (12) is not true. On the one hand, we have that for any $\epsilon > 0$ there is an $s \in (0, 1)$ such that

$$(14) \quad \frac{\left(\frac{2\pi}{|J(a)|}\right) \int_{J(a)} \rho(\phi \circ \sigma_b(\xi), \phi \circ \sigma_b(a))^2 |d\xi|}{(1 - |\phi \circ \sigma_b(a)|^2)^{1-p} (1 - |\sigma_b(a)|^2)^{q-1}} < \epsilon \quad \forall |\phi \circ \sigma_b(a)| > s.$$

Here we have used the pseudo-hyperbolic distance $\rho(z, w) = |\sigma_w(z)|$ between $z, w \in \mathbb{D}$ and the following basic estimate

$$\begin{cases} \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|_2^2 = \int_{\mathbb{T}} \rho(\phi(\xi), \phi(a))^2 P_a(\xi) |d\xi|; \\ P_a(\xi) = |\sigma'_a(\xi)| \geq 2^{-1}\pi |J(a)|^{-1} \quad \forall \xi \in J(a). \end{cases}$$

On the other hand, we can select two constants $s_0 \in (0, 1)$ and $\epsilon_0 > 0$, points $b_j \in \mathbb{D}$, and numbers $t_j \in (0, 1)$ with $\lim_{j \rightarrow \infty} t_j = 1$ such that for any $j = 1, 2, \dots$ one has $|\phi(b_j)| \leq s_0$ and

$$E_j = \{\xi \in \mathbb{T} : \phi_j(\xi) = \phi \circ \sigma_{b_j}(\xi) \text{ exists as radial limit and } |\phi_j(\xi)| > t_j\}$$

obeys

$$(15) \quad \left(\frac{(1 - |b_j|^2)^{1-q}}{(1 - |\phi(b_j)|^2)^{1-p}} \right) (2\pi)^{-1} |E_j| \geq \epsilon_0.$$

This (15), plus the above-stated lemma on (13), ensures that one can choose sets $F_j \subseteq E_j$ such that $|F_j| > 0$ and

$$(16) \quad \begin{aligned} & \left(\frac{(1 - |b_j|^2)^{1-q}}{(1 - |\phi(b_j)|^2)^{1-p}} \right) \frac{|J(r\xi) \cap E_j|}{|J(r\xi)|} \\ & \geq \left(\frac{(1 - |b_j|^2)^{1-q}}{(1 - |\phi(b_j)|^2)^{1-p}} \right) \left(\frac{|E_j|}{2^4\pi} \right) \\ & \geq 2^{-3}\epsilon_0 \quad \forall r \in [0, 1) \text{ \& } \xi \in F_j. \end{aligned}$$

If $\epsilon = 2^{-4}\epsilon_0$ in (14), then one can take such s that $s_0 < s < 1$ and (14) is true for $|\phi \circ \sigma_b(a)| > s$. Assuming $t_j \geq s$ and recalling that the definition of E_j ensures

$$|\phi \circ \sigma_{b_j}(r\xi)| \rightarrow |\phi \circ \sigma_{b_j}(\xi)| > t_j \quad \text{as } r \rightarrow 1 \text{ for each } \xi \in E_j.$$

Of course, this last property is valid for arbitrarily chosen point $\xi_j \in F_j$. Note that $|\phi \circ \sigma_{b_j}(0)| = |\phi(b_j)| \leq s_0$. Thus, by continuity of $|\phi \circ \sigma_{b_j}|$ there exists an $r_j \in (0, 1)$ such that $|\phi \circ \sigma_{b_j}(r_j\xi_j)| = s$. If $a_j = r_j\xi_j$ then

$$\rho(\phi \circ \sigma_{b_j}(\xi), \phi \circ \sigma_{b_j}(a_j)) \geq \rho(t_j, s) \quad \forall \quad \xi \in E_j,$$

and hence (16) and $q = 1$ are applied to deduce

$$\begin{aligned} & \frac{(1 - |\sigma_{b_j}(a_j)|^2)^{1-q}}{(1 - |\phi \circ \sigma_{b_j}(a_j)|^2)^{1-p}} \int_{J(a_j)} \rho(\phi \circ \sigma_{b_j}(\xi), \phi \circ \sigma_{b_j}(a_j))^2 \frac{|d\xi|}{(2\pi)^{-1}|J(a_j)|} \\ & \geq \left(\frac{(1 - |b_j|^2)^{1-q}}{(1 - |\phi(b_j)|^2)^{1-p}} \right) \left(\frac{|J(a_j) \cap E_j|}{|J(a_j)|} \right) \left(\frac{1 - |\phi(b_j)|^2}{1 - |\phi \circ \sigma_{b_j}(a_j)|^2} \right)^{1-p} \rho(t_j, s)^2 \\ & \geq 2^{-3}\epsilon_0 \left(\frac{\min\{1, (1 - s_0^2)^{1-p}\}}{(1 - s^2)^{1-p}} \right) \rho(t_j, s)^2. \end{aligned}$$

Since $\lim_{j \rightarrow \infty} \rho(t_j, s) = 1$, it follows from (14) that

$$\begin{aligned} & 0 \\ & = \lim_{j \rightarrow \infty} \frac{(1 - |\sigma_{b_j}(a_j)|^2)^{1-q}}{(1 - |\phi \circ \sigma_{b_j}(a_j)|^2)^{1-p}} \int_{J(a_j)} \left(\frac{\rho(\phi \circ \sigma_{a_j}(\xi), \phi \circ \sigma_{a_j}(b_j))^2}{(2\pi)^{-1}|J(a_j)|} \right) |d\xi| \\ & \geq 2^{-3}\epsilon_0 \left(\frac{\min\{1, (1 - s_0^2)^{1-p}\}}{(1 - s^2)^{1-p}} \right), \end{aligned}$$

a contradiction. In other words, (12) must be true under (4) being valid. \square

Proof of Theorem 2. Suppose that $C_\phi : C\mathcal{A}_p \mapsto C\mathcal{A}_q$ is compact. Of course, this operator is bounded, and thus (1) holds. Choosing $b = \phi(a)$, we see that f_b defined in Lemma 1 tends to 0 uniformly on compact subsets of \mathbb{D} whenever $|b| \rightarrow 1$. Thus, $\lim_{|b| \rightarrow 1} \|C_\phi f_b\|_{C\mathcal{A}_{p,*}} = 0$. As an immediate by-product of the C-part in the proof of Theorem 1, we have

$$0 = \lim_{|b| \rightarrow 1} \|C_\phi f_b\|_{C\mathcal{A}_{p,*}}^2 \gtrsim \lim_{|b| \rightarrow 1} \frac{(1 - |a|^2)^{1-q}|b|^2}{(1 - |b|^2)^{1-p}} \|\sigma_b \circ \phi \circ \sigma_a\|_2^2,$$

whence deriving (4).

Next, we deal with the converse part of Theorem 2 according to $(p, q) \in [0, 2) \times [1, 1]$ and $(p, q) \in (1, 2) \times [0, 2)$. In order to verify that $C_\phi : C\mathcal{A}_p \mapsto C\mathcal{A}_q$ is a compact operator, it suffices to check that $\lim_{n \rightarrow \infty} \|C_\phi f_n\|_{C\mathcal{A}_{q,*}} = 0$ holds for any sequence $(f_n)_{n=1}^\infty$ in $C\mathcal{A}_p$ with $\|f_n\|_{C\mathcal{A}_{p,*}} \leq 1$ and $f_n \rightarrow 0$ on compact subsets of \mathbb{D} as $n \rightarrow \infty$.

Situation I - assume that (1) holds and (4) is valid for $(p, q) \in [0, 2) \times [1, 1]$. Upon writing

$$\|C_\phi f_n\|_{\mathcal{CA}_{q,*}}^2 \lesssim \sup_{|\phi(a)| > s} T(n, a, q) + \sup_{|\phi(a)| \leq s} T(n, a, q),$$

where

$$0 < s < 1 \text{ \& } T(n, a, q) = (1 - |a|^2)^{1-q} \|f_n \circ \phi \circ \sigma_a - f_n \circ \phi(a)\|_2^2,$$

we have to control $\sup_{|\phi(a)| > s} T(n, a, q)$ and $\sup_{|\phi(a)| \leq s} T(n, a, q)$ from above. To do so, set

$$\begin{cases} f_{n,a} = f_n \circ \phi \circ \sigma_a - f_n(\phi(a)); \\ g_{n,a} = f_n \circ \sigma_{\phi(a)} - f_n(\phi(a)); \\ \psi_a = \sigma_{\phi(a)} \circ \phi \circ \sigma_a; \\ E(\phi, a, t) = \{\xi \in \mathbb{T} : |\sigma_{\phi(a)} \circ \phi \circ \sigma_a(\xi)| > t\}. \end{cases}$$

Using (9) we obtain

$$\begin{aligned} & \sup_{|\phi(a)| > s} T(n, a, q) \\ & \approx \sup_{|\phi(a)| > s} (1 - |a|^2)^{1-q} \|f_{n,a}\|_2^2 \\ & \lesssim \sup_{|\phi(a)| > s} (1 - |a|^2)^{1-q} \|g_{n,a}\|_2^2 \|\psi_a\|_2^2 \\ & \lesssim \sup_{|\phi(a)| > s} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\psi_a\|_2^2 \|f_n\|_{\mathcal{CA}_{p,*}}^2 \\ & \lesssim \sup_{|\phi(a)| > s} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\psi_a\|_2^2, \end{aligned}$$

whence getting by (4)

$$(17) \quad \lim_{s \rightarrow 1} \sup_{|\phi(a)| > s} T(n, a, q) = 0 \quad \forall \quad n = 1, 2, 3, \dots$$

Meanwhile,

$$\sup_{|\phi(a)| \leq s} T(n, a, q) \lesssim \sup_{|\phi(a)| \leq s} T_1(n, a, q) + \sup_{|\phi(a)| \leq s} T_2(n, a, q),$$

where

$$\begin{cases} T_1(n, a, q) = (1 - |a|^2)^{1-q} \int_{\mathbb{T} \setminus E(\phi, a, t)} |f_{n,a}(\xi)|^2 |d\xi|; \\ T_2(n, a, q) = (1 - |a|^2)^{1-q} \int_{E(\phi, a, t)} |f_{n,a}(\xi)|^2 |d\xi|. \end{cases}$$

Applying Schwarz's lemma to $g_{n,a}$ or using [6, (3.19)] we get

$$\sup_{|z| \leq t} |z|^{-1} |g_{n,a}(z)| \leq 2 \sup_{|w| \leq t} |g_{n,a}(w)|$$

thereby deriving

$$\begin{aligned}
& \sup_{|\phi(a)| \leq s} T_1(n, a, q) \\
& \lesssim \sup_{|\phi(a)| \leq s} (1 - |a|^2)^{1-q} \sup_{|w| \leq t} |g_{n,a}(w)|^2 \int_{\mathbb{T}} |\psi_a(\xi)|^2 |d\xi| \\
& \lesssim (1 + (1 - s)^{p-1}) \sup_{|w| \leq t} |g_{n,a}(w)|^2 \sup_{|\phi(a)| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\psi_a\|_2^2 \\
& \rightarrow 0 \quad \text{as } n \rightarrow \infty,
\end{aligned}$$

in which $|\phi(a)| \leq s$ and $|w| \leq t$ have been used. Also, a combination of (9), (1) and $q = 1$ gives that if

$$\begin{cases} \lambda = (a\bar{b} - 1)/(1 - b\bar{a}); \\ \tau = \phi \circ \sigma_a; \\ c = \sigma_b(a); \\ b \in \mathbb{D}, \end{cases}$$

then

$$\begin{aligned}
& \|f_{n,a}\|_{\mathcal{CA}_{q,*}}^2 \\
& = \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-q} \|f_n \circ \tau \circ \sigma_b - f_n \circ \tau(b)\|_2^2 \\
& \lesssim \sup_{b \in \mathbb{D}} (1 - |b|^2)^{1-q} \|f_n \circ \sigma_{\tau(b)} - f_n \circ \tau(b)\|_2^2 \|\sigma_{\tau(b)} \circ \tau \circ \sigma_b\|_2^2 \\
& \lesssim \|f_n\|_{\mathcal{CA}_{p,*}}^2 \sup_{b \in \mathbb{D}} \frac{(1 - |b|^2)^{1-q}}{(1 - |\tau(b)|^2)^{1-p}} \|\sigma_{\tau(b)} \circ \tau \circ \sigma_b\|_2^2 \\
& \lesssim \sup_{c \in \mathbb{D}} \frac{(1 - |c|^2)^{1-q}}{(1 - |\phi(c)|^2)^{1-p}} \|\sigma_{\phi(c)} \circ \phi \circ (\lambda \sigma_c)\|_2^2 \\
& \lesssim \sup_{c \in \mathbb{D}} \frac{(1 - |c|^2)^{1-q}}{(1 - |\phi(c)|^2)^{1-p}} \|\psi_c\|_2^2 \\
& < \infty,
\end{aligned}$$

and hence from the Cauchy-Schwarz inequality, Lemmas 3-4 and $q = 1$ it follows that

$$\begin{aligned}
& \sup_{|\phi(a)| \leq s} T_2(n, a, q) \\
& \lesssim \sup_{|\phi(a)| \leq s} (1 - |a|^2)^{1-q} \left(\int_{E(\phi, a, t)} |f_{n,a}(\xi)|^4 |d\xi| \right)^{\frac{1}{2}} |E(\phi, a, t)|^{\frac{1}{2}} \\
& \lesssim \sup_{|\phi(a)| \leq s} \left(\frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{p-1}} \int_{\mathbb{T}} |f_{n,a}(\xi)|^4 |d\xi| \right)^{\frac{1}{2}} \left(\frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} |E(\phi, a, t)| \right)^{\frac{1}{2}} \\
& \lesssim (1 + (1 - s^2)^{1-p})^{\frac{1}{2}} \sup_{|\phi(a)| \leq s} \left(\frac{\|f_{n,a}\|_2^2 \|f_{n,a}\|_{C\mathcal{A}_{q,*}}^2}{(1 - |a|^2)^{q-1}} \right)^{\frac{1}{2}} \left(\frac{|E(\phi, a, t)|}{\frac{(1 - |\phi(a)|^2)^{1-p}}{(1 - |a|^2)^{1-q}}} \right)^{\frac{1}{2}} \\
& \lesssim (1 + (1 - s^2)^{1-p})^{\frac{1}{2}} \|f_{n,a}\|_{C\mathcal{A}_{q,*}}^2 \sup_{|\phi(a)| \leq s} \left(\frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} |E(\phi, a, t)| \right)^{\frac{1}{2}} \\
& \lesssim \left(\sup_{c \in \mathbb{D}} \frac{(1 - |c|^2)^{1-q}}{(1 - |\phi(c)|^2)^{1-p}} \|\psi_c\|_2^2 \right) \sup_{|\phi(a)| \leq s} \left(\frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} |E(\phi, a, t)| \right)^{\frac{1}{2}} \\
& \rightarrow 0 \quad \text{as } t \rightarrow 1.
\end{aligned}$$

Consequently,

$$(18) \quad \lim_{n \rightarrow \infty} \sup_{|\phi(a)| \leq s} T(n, a, q) = 0.$$

Putting (17) and (18) together, we reach $\lim_{n \rightarrow \infty} \|C_\phi f_n\|_{C\mathcal{A}_{q,*}} = 0$.

Situation 2 - assume that (1) holds and (4) is valid for $(p, q) \in (1, 2) \times [0, 2)$. Rewriting

$$\begin{aligned}
& \|C_\phi f_n\|_{C\mathcal{A}_{q,*}}^2 \\
& \lesssim \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-q} \int_{\mathbb{D}} |f'_n(w)|^2 N(\phi \circ \sigma_a, w) dm(w) \\
& \leq \sup_{a \in \mathbb{D}} U(n, a, q, r) + \sup_{a \in \mathbb{D}} V(n, a, q, r),
\end{aligned}$$

where $2^{-1} \leq r < 1$ and

$$\begin{cases} U(n, a, q, r) = (1 - |a|^2)^{1-q} \int_{|\sigma_{\phi(a)}(w)| \leq r} |f'_n(w)|^2 N(\phi \circ \sigma_a, w) dm(w); \\ V(n, a, q, r) = (1 - |a|^2)^{1-q} \int_{|\sigma_{\phi(a)}(w)| > r} |f'_n(w)|^2 N(\phi \circ \sigma_a, w) dm(w), \end{cases}$$

we have to control $\sup_{a \in \mathbb{D}} U(n, a, q, r)$ and $\sup_{a \in \mathbb{D}} V(n, a, q, r)$ for an appropriate $r \in [2^{-1}, 1)$. In the sequel, let $b = \phi(a)$.

Sub-situation 1 - estimate for $\sup_{a \in \mathbb{D}} U(n, a, q, r)$. For this, we consider two cases for any given $s \in (0, 1)$.

Case 1₁: $|b| \leq s$. Under this case, $|\sigma_b(w)| \leq r$ ensures that w belongs to a compact subset K of \mathbb{D} , and therefore, it follows from $f_n \rightarrow 0$ on any compact subset of \mathbb{D} and (11) that $\lim_{n \rightarrow \infty} \sup_{w \in K} |f'_n(w)| = 0$ and consequently,

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \sup_{|b| \leq s} (1 - |a|^2)^{1-q} \int_{|\sigma_b(w)| \leq r} |f'_n(w)|^2 N(\phi \circ \sigma_a, w) dm(w) \\
&= \lim_{n \rightarrow \infty} \sup_{|b| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} (1 - |b|^2)^{p-1} \int_{|\sigma_b(w)| \leq r} |f'_n(w)|^2 N(\phi \circ \sigma_a, w) dm(w) \\
&\lesssim \left(\lim_{n \rightarrow \infty} \sup_{w \in K} |f'_n(w)|^2 \right) \sup_{|b| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \int_{|\sigma_b(w)| \leq r} N(\phi \circ \sigma_a, w) dm(w) \\
&\lesssim \left(\lim_{n \rightarrow \infty} \sup_{w \in K} |f'_n(w)|^2 \right) \sup_{|b| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \int_{\mathbb{D}} N(\sigma_b \circ \phi \circ \sigma_a, z) dm(z) \\
&\lesssim \left(\lim_{n \rightarrow \infty} \sup_{w \in K} |f'_n(w)|^2 \right) \sup_{|b| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \|\sigma_b \circ \phi \circ \sigma_a\|_2^2 \\
&= 0.
\end{aligned}$$

Case 1₂: $|b| > s$. Using (8) we get

$$\begin{aligned}
& \sup_{|b| > s} (1 - |a|^2)^{1-q} \int_{|\sigma_b(w)| \leq r} |f'_n(w)|^2 N(\phi \circ \sigma_a, w) dm(w) \\
&\lesssim \|f_n\|_{\mathcal{CA}_{p,*}}^2 \sup_{|b| > s} (1 - |a|^2)^{1-q} \int_{|\sigma_b(w)| \leq r} N(\sigma_b \circ \phi \circ \sigma_a, \sigma_b(w)) \frac{dm(w)}{(1 - |w|^2)^{3-p}} \\
&\lesssim \sup_{|b| > s} (1 - |a|^2)^{1-q} \int_{|z| \leq r} (1 - |\sigma_b(z)|^2)^{p-1} N(\sigma_b \circ \phi \circ \sigma_a, z) \frac{dm(z)}{(1 - |z|^2)^2} \\
&\lesssim \sup_{|b| > s} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \int_{|z| \leq r} N(\sigma_b \circ \phi \circ \sigma_a, z) \frac{dm(z)}{(1 - |z|^2)^2} \\
&\lesssim (1 - r^2)^{-2} \sup_{|b| > s} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \|\sigma_b \circ \phi \circ \sigma_a\|_2^2 \\
&\rightarrow 0 \quad \text{as } s \rightarrow 1.
\end{aligned}$$

Putting the above two cases together, we see that for any $\epsilon \in (0, 1)$ there are two real numbers: $r_0 \in [2^{-1}, 1)$; $s_0 \in (0, 1)$, and a natural number n_0 such that $n \geq n_0$

$$(19) \quad \sup_{a \in \mathbb{D}} U(n, a, q, r_0) \leq \sup_{|b| \leq s_0} U(n, a, q, r_0) + \sup_{|b| > s_0} U(n, a, q, r_0) < \epsilon.$$

Sub-situation 2 - estimate for $\sup_{a \in \mathbb{D}} V(n, a, q, r)$. Like *Sub-situation 1*, two treatments are required.

Case 2₁: $|b| \leq s$. For this case, we need the following by-product of [15, Lemma 2.1]: if ψ is an analytic self-map of \mathbb{D} with $\psi(0) = 0$ then

$$(20) \quad W = \sup_{0 < |w| < 1} |w|^2 N(\psi, w) < \infty \implies \sup_{2^{-1} \leq |w| < 1} \frac{N(\psi, w)}{\ln |w|^{-1}} \leq 4(\ln 2)^{-1} W.$$

Note that (1) and (4) imply respectively

$$(21) \quad \sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \sup_{0 < |w| < 1} |w|^2 N(\sigma_b \circ \phi \circ \sigma_a, w) < \infty$$

and

$$(22) \quad \lim_{|b| \rightarrow 1} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \sup_{0 < |w| < 1} |w|^2 N(\sigma_b \circ \phi \circ \sigma_a, w) = 0$$

thanks to the following (10)-based mean value estimate for $N(\sigma_w \circ \sigma_b \circ \phi \circ \sigma_a, 0)$ where $0 < |w| < 1$ (cf. [6, (2.9)]):

$$\begin{aligned} & |w|^2 N(\sigma_b \circ \phi \circ \sigma_a, w) \\ &= |w|^2 N(\sigma_w \circ \sigma_b \circ \phi \circ \sigma_a, 0) \\ &\lesssim \int_{|z| < |w|} N(\sigma_w \circ \sigma_b \circ \phi \circ \sigma_a, z) dm(z) \\ &\lesssim \int_{\mathbb{D}} N(\sigma_b \circ \phi \circ \sigma_a, \sigma_w(z)) dm(z) \\ &\approx \int_{\mathbb{D}} |\sigma'_w(z)|^2 N(\sigma_b \circ \phi \circ \sigma_a, z) dm(z) \\ &\approx \|\sigma_w \circ \sigma_b \circ \phi \circ \sigma_a - \sigma_w \circ \sigma_b \circ \phi \circ \sigma_a(0)\|_2^2 \\ &\lesssim \|\sigma_b \circ \phi \circ \sigma_a\|_2^2. \end{aligned}$$

Thus, a combination of (20)-(21)-(22) and Hölder's inequality gives

$$\begin{aligned}
& \sup_{|b| \leq s} (1 - |a|^2)^{1-q} \int_{|\sigma_b(w)| > r} |f'_n(w)|^2 N(\phi \circ \sigma_a, w) dm(w) \\
& \approx \sup_{|b| \leq s} (1 - |a|^2)^{1-q} \int_{|\sigma_b(w)| > r} |f'_n(w)|^2 N(\sigma_b \circ \phi \circ \sigma_a, \sigma_b(w)) dm(w) \\
& \lesssim \sup_{|b| \leq s} (1 - |b|^2)^{1-p} \int_{|\sigma_b(w)| > r} |f'_n(w)|^2 N(\sigma_b, w) dm(w) \\
& \lesssim (1 + (1 - s^2)^{1-p}) \int_{|\sigma_b(w)| > r} |f'_n(w)|^2 N(\sigma_b, w) dm(w) \\
& \lesssim (1 + (1 - s^2)^{1-p}) \int_{|z| > r} |(f_n \circ \sigma_b)'(z)|^2 (1 - |z|^2) dm(z) \\
& \lesssim (1 + (1 - s^2)^{1-p}) \left(\frac{\int_{|z| > r} |(f_n \circ \sigma_b)'(z)|^4 (1 - |z|^2)^{4-p} dm(z)}{\left(\int_{|z| > r} (1 - |z|^2)^{p-2} dm(z) \right)^{-1}} \right)^{1/2}.
\end{aligned}$$

Since $\|f_n\|_{C\mathcal{A}_{p,*}} \leq 1$ and $|b| \leq s < 1$ ensure $\|f_n \circ \sigma_b\|_{C\mathcal{A}_{p,*}} \lesssim 1$, one concludes that $|(f_n \circ \sigma_b)'(z)|^2 (1 - |z|^2) dm(z)$ is p -Carleson measure with norm relying on s and so that $d\mu_n(z) = |(f_n \circ \sigma_b)'(z)|^2 (1 - |z|^2)^{4-p} dm(z)$ is 3-Carleson measure with norm relying on s . Now, it follows from [16, Theorem 1.2] that

$$\begin{aligned}
& \int_{|z| > r} |(f_n \circ \sigma_b)'(z)|^4 (1 - |z|^2)^{4-p} dm(z) \\
& = \int_{|z| > r} |(f_n \circ \sigma_b)'(z)|^2 d\mu_n(z) \\
& \lesssim \|\mu_n\|_{CM_3} \int_{\mathbb{D}} |(f_n \circ \sigma_b)'(z)|^2 (1 - |z|^2) dm(z) \\
& \lesssim \|f_n\|_{C\mathcal{A}_{p,*}}^4 \\
& \lesssim 1.
\end{aligned}$$

Note that

$$\lim_{r \rightarrow 1} \int_{|z| > r} (1 - |z|^2)^{p-2} dm(z) = 0.$$

So

$$\lim_{r \rightarrow 1} \sup_{|b| \leq s} (1 - |a|^2)^{1-q} \int_{|\sigma_b(w)| > r} |f'_n(w)|^2 N(\phi \circ \sigma_a, w) dm(w) = 0$$

holds for any $n = 1, 2, 3, \dots$

Case 2₂: $|b| > s$. Since (22) implies that for any $\epsilon \in (0, 1)$ there is an $s_0 \in (0, 1)$ such that

$$|b| > s_0 \implies \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \sup_{0 < |w| < 1} |w|^2 N(\sigma_b \circ \phi \circ \sigma_a, w) < \epsilon.$$

Thus, (20) is applied once again to deduce that

$$\begin{aligned} & N(\phi \circ \sigma_a, w) \\ &= N(\sigma_b \circ \phi \circ \sigma_a, \sigma_b(w)) \\ &\lesssim \frac{\epsilon(1 - |b|^2)^{1-p}}{(1 - |a|^2)^{1-q}} \ln |\sigma_b(w)|^{-1} \\ &\approx \frac{\epsilon(1 - |b|^2)^{1-p}}{(1 - |a|^2)^{1-q}} N(\sigma_b, w) \\ &\text{as } |\sigma_b(w)| > r > 2^{-1}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \sup_{|b| > s_0} (1 - |a|^2)^{1-q} \int_{|\sigma_b(w)| > r} |f'_n(w)|^2 N(\phi \circ \sigma_a, w) dm(w) \\ &\lesssim \epsilon \sup_{|b| > s_0} (1 - |b|^2)^{1-p} \int_{|\sigma_b(w)| > r} |f'_n(w)|^2 N(\sigma_b, w) dm(w) \\ &\lesssim \epsilon \|f_n\|_{C\mathcal{A}_{p,*}}^2 \\ &\lesssim \epsilon. \end{aligned}$$

The previous discussions on *Case 2₁* and *Case 2₂* indicate

$$(23) \quad \lim_{n \rightarrow \infty} \sup_{a \in \mathbb{D}} V(n, a, q, r) = 0.$$

Obviously, (19) and (23) give $\lim_{n \rightarrow \infty} \|C_\phi f_n\|_{C\mathcal{A}_{q,*}} = 0$. □

Proof of Corollary 2. This follows from (7), Theorem 2 and [20, Theorem 1.4 (c)]. □

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